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WEAK MIXING AND UNIQUE ERGODICITY ON HOMOGENEOUS SPACES

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ABSTRACT

Under certain conditions the weak mixing of a translation on G/Γ implies that the action of an associated subgroup of G on G/Γ is uniquely ergodic. This result generalizes earlier theorems of Furstenberg and Veech.

Let G be a Lie group and Γ a discrete subgroup with G/Γ compact. Then G acts on G/Γ by $T_g(h\Gamma) = gh\Gamma$. We will assume G unimodular so that some Haar measure on G induces a G-invariant probability measure μ on G/Γ. On G/Γ one takes a Riemannian metric induced from a right invariant metric on G. For $a \in G$ the automorphism $\mathrm{Ad}_a(g) = aga^{-1}$ of G corresponds to an automorphism ad_a of the Lie algebra of G. Let $\mathfrak{G}^s(a)$, $\mathfrak{G}^c(a)$, $\mathfrak{G}^u(a)$ denote the invariant subspaces of $\mathfrak{G}$ corresponding to the eigenvalues of ad_a satisfying $|\lambda| < 1$, $|\lambda| = 1$, $|\lambda| > 1$ respectively. These subspaces are subalgebras of $\mathfrak{G}$; let G^u be the subgroup of G corresponding to $\mathfrak{G}^u(a)$. In this paper we prove the

THEOREM. *Assume that T_a is weak mixing on $(G/\Gamma, \mu)$ and that $\mathrm{ad}_a \mid \mathfrak{G}^c(a)$ is a semisimple linear map. Then action of G^u on G/Γ on the left is uniquely ergodic.*

This means that μ is the only Borel probability measure on G/Γ invariant under G^u. One case of the above is a theorem of Furstenberg [3]; here T_a is (part of) a geodesic flow and G^u is a horocycle flow. In Section 3 one will see that this theorem also includes Veech's [11] generalization of Furstenberg to semi-simple Lie groups. The basic idea used in the present paper occurs in [2]; in some sense one can find it in [1] and [11]. The papers [2] and [6] give analogues of Furstenberg's theorem in dynamical systems which move away from the Lie group framework.

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1. Proof of the Theorem

Let $\mathfrak{G}_C$ be the complexification of $\mathfrak{G}$ and $W_\lambda = \{v \in \mathfrak{G}_C : (\mathrm{ad}_a - \lambda I)^n v = 0$ some $n > 0\}$. Then $\mathfrak{G}_C = \bigoplus_{i=1}^m W_{\lambda_i}$ where $\lambda_1, \cdots, \lambda_m$ are the distinct eigenvalues of ad_a and also $[W_{\lambda_i}, W_{\lambda_j}] \subset W_{\lambda_i \lambda_j}$ (this standard fact follows as ad_a is a Lie algebra automorphism). From this it follows that

$$\mathfrak{G}_C^s = \bigoplus_{|\lambda_i|<1} W_{\lambda_i}, \mathfrak{G}_C^c = \bigoplus_{|\lambda_i|=1} W_{\lambda_i}, \mathfrak{G}_C^u = \bigoplus_{|\lambda_i|>1} W_{\lambda_i}$$

are subalgebras of $\mathfrak{G}_C$. These are the complexifications of $\mathfrak{G}^s(a)$, $\mathfrak{G}^c(a)$, $\mathfrak{G}^u(a)$, which are therefore subalgebras of $\mathfrak{G}$ as we claimed earlier. Assuming that $|\lambda_1| \leqq \cdots \leqq |\lambda_k|$ are the eigenvalues with $|\lambda_i| > 1$, one has $[\mathfrak{G}_C^u, \bigoplus_{i=j}^k W_{\lambda_i}] \subset \bigoplus_{i=j+1}^k W_{\lambda_i}$. From this it follows that $\mathfrak{G}_c^u$ and hence $\mathfrak{G}^u(a)$ is nilpotent.

From $[W_{\lambda_i}, W_{\lambda_j}] \subset W_{\lambda_i \lambda_j}$ one sees that $\mathfrak{G}^{cs}(a) = \mathfrak{G}^c(a) + \mathfrak{G}^s(a)$ is a subalgebra of $\mathfrak{G}$; let G^{cs} be the corresponding subgroup of G. One defines two C^∞ foliations on G/Γ by

$$\mathcal{G}^u = \{G^u x\}_{x \in G/\Gamma} \text{ and } \mathcal{G}^{cs} = \{G^{cs} x\}_{x \in G/\Gamma}.$$

These foliations are invariant under T_a in the sense that

$$T_a(G^u x) = (aG^u a^{-1})ax = G^u T_a x \text{ and } T_a(G^{cs} x) = G^{cs} T_a x.$$

The two foliations $\mathcal{G}^u$ and $\mathcal{G}^{cs}$ are transversal with complementary dimensions. Let $p = \dim G^u$ and $q = \dim G^{cs}$; pick small disk neighborhoods $U \subset G^{cs}$ and $V \subset G^u$ of e. One can find open sets $M_1, \cdots, M_r$ covering G/Γ and C^∞ diffeomorphisms $\phi_i = (\phi_i^1, \phi_i^2): M_i \to D^p \times D^q$ (D^n is the n-disk) so that

$$\phi_i(M_i \cap Vx) = D^p \times \phi_i^2(x) \text{ and } \phi_i(M_i \cap Ux) = \phi_i^1(x) \times D^q$$

for every $x \in M_i$.

On a leaf $G^u x$ of $\mathcal{G}^u$ we let μ^u denote the measure induced from the Riemannian metric of G/Γ restricted to $G^u x$. Since the Riemannian metric on G/Γ came from a right invariant metric on G, we see that μ^u comes from a right Haar measure μ^* on G^u via the immersion $G^u \to G^u x$ (by $g \to gx$).

LEMMA. *There is a function $\alpha(\delta) > 0$ defined for $\delta > 0$ with* $\lim_{\delta \to 0} \alpha(\delta) = 1$ *and for which the following holds: If L_1, L_2 are compact subsets of $G^u x_1$, $G^u x_2$ and there is a continuous function $k: L_1 \to B_\delta(e, G^{cs})$ and a bijection $h: L_1 \to L_2$ so that $h(z) = k(z)z$, then $\mu^u(L_1) \geqq \alpha(\delta)\mu^u(L_2)$.*

PROOF. Let $M_i^* \subset M_i$ be compact with $G/\Gamma \subset \bigcup_{i=1}^r \mathrm{int}\, M_i^*$. It is enough to prove the statement for L_1, $L_2 \subset$ some M_i^* and $\phi_i(L_j) = L_j' \times u_j$. Then $L_1' = L_2'$

and the $\phi_i(L_j)$ have the same measure using the Euclidean metric. Since the ϕ_i: $M_i^* \to R^{p+q}$ are C^∞ and the M_i^* are compact, the ratio $\mu^u(L_1)/\mu^u(L_2)$ is bounded and tends to 1 as $\delta \to 0$ (i.e., $\|u_1 - u_2\| \to 0$). ■

Let ν denote any G^u-invariant Borel probability measure.

LEMMA 2. *Suppose $R \subset G^u$ is open, $S \subset G^{cs}x$ is compact, and $(r, s) \to rs$ is one-to-one on $R \times S \to G/\Gamma$. If $X \subset Y \subset RS$ are Borel sets, then $\nu(X)/\nu(Y) \in [\inf \Phi, \sup \Phi]$ where*

$$\Phi = \left\{ \frac{\mu^u(X \cap Rs)}{\mu^u(Y \cap Rs)} : s \in S, \mu^u(Y \cap Rs) > 0 \right\}.$$

PROOF. Define m on S by $m(E) = \nu(RE)$ for $E \subset S$. There are Borel measures ν_s on Rs for m almost every $s \in S$ such that [9]:

$$\nu(W) = \int \nu_s(W \cap Rs) dm(s)$$

for $W \subset RS$ Borel. The measures ν_s are determined (for m-almost all s) by this formula. As $\nu(gW) = \nu(W)$ for $g \in G^u$, this uniqueness implies that for m-almost all s the measure ν_s comes from a left (local) Haar measure on $R \subset G^u$ via the identification $R \to Rs$. Since G^u is unimodular (it is nilpotent), ν_s is proportional to $\mu^u \mid Rs$ for m-almost all s. The lemma follows by applying the formula to both $W = X$ and $W = Y$. ■

We call a Borel set $H \subset G/\Gamma$ of small diameter a *box* if for any points $x_1, x_2 \in H$ the point $Ux_1 \cap Vx_2 \in H$. A set of small diameter H is contained in some M_i; H is seen to be a box if and only if $\phi_i(H) = L \times K$ for some Borel sets $L \subset D^p$ and $K \subset D^q$.

LEMMA 3. *For any $\beta > 0$ there is a finite cover of G/Γ by boxes $H_1, \cdots, H_t$ of diameter at most β with $\mu(H_j) > 0$ and $\nu(a^n H_j \cap a^n H_k) = 0$ for $j \neq k$, $n \geqq 0$.*

PROOF. Define the measure ν_n by $\nu_n(E) = \nu(a^n E)$. For $g \in G^u$ one has

$$\nu_n(gE) = \nu(a^n gE) = \nu((a^n g a^{-n}) a^n E) = \nu(a^n E) = \nu_n(E).$$

So $\nu' = \Sigma_{n=0}^{\infty} (1/2^n) \nu_n$ is a G^u-invariant measure. It is enough to cover G/Γ by small boxes $H_1, \cdots, H_t$ with pairwise disjoint interiors, $H_j = \overline{\operatorname{int} H_j}$, and $\nu'(\partial H_j) = 0$. Lemma 2 applied to ω' in place of ν gives us a measure m_i' on $S_i = \phi_i^{-1}(0 \times D^q)$ when we use $R = V$. We identify S_i with $0 \times D^q$ and think of m_i' as being on $0 \times D^q$. For a box $H \subset M_i$ let us write $\phi_i(H) = L \times K$; then $\omega'(\partial H) = 0$ provided

$$(*) \qquad m_i'(\partial K) = 0 \text{ and } \mu''(\partial L) = 0.$$

We construct inductively on i a cover $\{H_1, \cdots, H_{t_i}\}$ of $M_1 \cup \cdots \cup M_i$ by boxes such that:

$$H_j = \overline{\text{int } H_j}, \text{ int } H_j \cap \text{int } H_k = \varnothing \text{ for } j \neq k,$$

$$\text{and } (*) \text{ holds for } H_j \text{ with } j \in (t_{i-1}, t_i].$$

For $j \in [1, t_i]$ let

$$\phi_{i+1}(M_{i+1} \cap \text{int } H_j) = L_j \times K_j.$$

One sees that (*) holding for the $i' \leqq i$ with $j \in (t_{i'-1}, t_{i'}]$ implies that

$$m'_{i+1}(\partial K_j) = 0 \text{ and } \mu''(\partial L_j) = 0.$$

For every subset $\Lambda \subset \{1, \cdots, t_i\}$ (including $\Lambda = \varnothing$) define

$$K_\Lambda = \left(D^q \bigcap_{j \in \Lambda} K_j\right) \setminus \bigcup_{j \notin \Lambda} \overline{K_j}$$

$$L_\Lambda = \left(D^p \bigcap_{j \in \Lambda} K_j\right) \setminus \bigcup_{j \notin \Lambda} \overline{L_j}.$$

Slightly shrinking M_{i+1} if necessary, we may assume $m'_{i+1}(\partial D^q) = 0$ and $\mu''(\partial D^p) = 0$. As $\partial K_\Lambda \subset \partial D^q \cup \bigcup_j \partial K_j$, one has $m'_{i+1}(\partial K \Lambda) = 0$. Also $K_\Lambda = K_{\Lambda'}$ if $K_\Lambda \cap K_{\Lambda'} \neq \varnothing$, and $D^q = \bigcup_\Lambda \bar{K}$. Similarly, $\mu''(\partial L \Lambda) = 0$, $L_\Lambda = L_{\Lambda'}$ if $L_\Lambda \cap L_{\Lambda'} \neq \varnothing$, and $D^p = \bigcup_\Lambda \bar{L}_\Lambda$. Let $H_{t_i+1}, \cdots, H_{t_{i+1}}$ be all the $\phi_{i+1}^{-1}(\overline{L_\Lambda \times K_\Lambda})$ not contained in some H_j with $j \in [1, t_i]$. ■

We will now show $\nu = \mu$. Let $F = \phi_i^{-1}(L \times K)$ be a compact box and let $F_\gamma = \phi_i^{-1}(L \times B_\gamma(K))$ where $B_\gamma(K)$ is the γ-neighborhood of K in R^q for $\gamma > 0$. There is a $\delta > 0$ so that $kx \in F_\gamma$ when $x \in F$ and $k \in B_\delta(e, G^{cs})$. Finally, since $\text{ad}_a \mid \mathfrak{G}^c(a)$ is semisimple and $\text{ad}_a \mid \mathfrak{G}^s(a)$ has all eigenvalues $|\lambda| < 1$, there is an $\varepsilon > 0$ so that

$$k \in B_\varepsilon(e, G^{cs}) \Rightarrow a^n k a^{-n} \in B_\delta(e, G^{cs}) \; n \geqq 0.$$

If $x' = kx$ with $x \in G/\Gamma$ and $k \in B_\varepsilon(e, G^{cs})$, then $T_a^n x' = (a^n k a^{-n}) T_a^n x$ and $a^n k a^{-n} \in B_\delta(e, G^{cs})$.

Let $H_1, \cdots, H_t$ be a cover of G/Γ by disjoint boxes given by Lemma 3 of such small diameter that

$$x \in H_j,\ x' \in Ux \cap H_j \Rightarrow x' = kx \text{ with } k \in B_\varepsilon(e, G^{cs}).$$

Pick $z_j \in H_j$. For $y \in H_j \cap Uz_j$ and $m > 0$ define

$$A_m(y) = Vy \cap H_j \cap a^{-m}F$$

and

$$A_m^\gamma(y) = Vy \cap H_j \cap a^{-m}F_\gamma.$$

For $y, y' \in H_j \cap Uz_j$ one has a one-to-one map $A_m(y) \to A_m^\gamma(y')$ defined by $x \to x' = Vy' \cap Ux$. For then $x' = kx$ with $k \in B_\varepsilon(e, G^{cs})$ and $a^m x' = (a^m k a^{-m}) a^m x$. Since $a^m x \in F$ we have $a^m x' \in F_\gamma$. Lemma 1 gives

$$\text{(I)} \qquad \mu^u(a^m A_m^\alpha(y')) \geqq \alpha(\delta) \mu^u(a^m A_m(y)).$$

Since T_a is weak mixing (see [4]) there is a sequence $m_k \to \infty$ so that

$$\lim_{k \to \infty} \mu(a^{m_k} H_j \cap F) = \mu(H_j)\mu(F)$$

for all $1 \leqq j \leqq t$. For large $m = m_k$ one has (using $\mu(a^m H_j) = \mu(H_j) > 0$)

$$\text{(II)} \qquad \frac{\mu(a^m H_j \cap F)}{\mu(a^m H_j)} \geqq (1 - \delta)\mu(F).$$

Now $a^m H_j \subset a^m (VUz_j) = (a^m V a^{-m})(a^m U z_j)$. We apply Lemma 2 to $x = a^m z_j$, $R = a^m V a^{-m}$, $S = (a^m U a^{-m}) a^m zg$, $X = a^m H_j \cap F$, $Y = a^m H_j$ and $\nu = \mu$. By (II) and Lemma 3 there is an $s \in a^m(H_j \cap Uz_j)$ with

$$\frac{\mu^u(a^m V a^{-m} s \cap a^m H_j \cap F)}{\mu^u(a^m V a^{-m} s \cap a^m H_j)} \geqq (1 - \delta)\mu(F).$$

Now $s = a^m y$ with $y \in H_j \cap Uz_j$ and one can rewrite this expression as

$$\text{(III)} \qquad \frac{\mu^u(a^m A_m(y))}{\mu^u(a^m(Vy \cap H_j))} \geqq (1 - \delta)\mu(F).$$

Using (I), for *any* $y' \in H_j \cap Uz_j$ one obtains

$$\text{(IV)} \qquad \frac{\mu^u(a^m A_m^\gamma(y'))}{\mu^u(a^m(Vy' \cap H_j))} \geqq \frac{\mu^u(a^m(Vy \cap H_j))}{\mu^u(a^m(Vy' \cap H_j))} \alpha(\delta)(1 - \delta)\mu(F)$$

$$\geqq \alpha(\delta)^2 (1 - \delta)\mu(F).$$

For the second inequality we use Lemma 1. Working backwards and applying Lemma 2 to ν, $X = a^m H_j \cap F_\gamma$ and $Y = a^m H_j$ we get

$$\frac{\nu(a^m H_j \cap F_\gamma)}{\nu(a^m H_j)} \geqq \alpha(\delta)^2(1-\delta)\mu(F).$$

This holds true for all large $m = m_k$ and all $1 \leqq j \leqq t$. Summing over j (using $\nu(a^m H_j \cap a^m H_k) = 0$)

$$\nu(F_\gamma) = \sum_j \nu(a^m H_j \cap F_\gamma) \geqq \alpha(\delta)^2(1-\delta)\mu(F).$$

Letting $\gamma \to 0$, $\delta \to 0$ and $\nu(F) \geqq \mu(F)$.

One obtains $\mu(F) \geqq \nu(F)$ analogously. Use a sequence $m_k \to \infty$ with $\mu(a^{m_k} H_j \cap F_\gamma) \to \mu(H_j)\mu(F_\gamma)$. For large $m = m_k$ one has

$$\text{(II')} \qquad (1+\delta)\mu(F_\gamma) \geqq \frac{\mu(a^m H_j \cap F_\gamma)}{\mu(a^m H_j)}.$$

Using Lemma 2 one finds a $y' \in H_j \cap Uz_j$ with

$$\text{(III')} \qquad (1+\delta)\mu(F_\gamma) \geqq \frac{\mu^u(a^m A_m{}^\gamma(y'))}{\mu^u(a^m(Vy' \cap H_j))}.$$

For any $y \in H_j \cap Uz_j$, (I) gives us

$$\text{(IV')} \qquad (1+\delta)\mu(F_\gamma)\alpha(\delta)^{-2} \geqq \frac{\mu^u(a^m A_m(y))}{\mu^u(a^m(Vy \cap H_j))}.$$

Then Lemma 3 gives

$$(1+\delta)\mu(F_\gamma)\alpha(\delta)^{-2} \geqq \frac{v(a^m H_j \cap F)}{v(a^m H_j)}$$

and so $(1+\delta)\mu(F_\gamma)\alpha(\delta)^{-2} \geqq \nu(F)$. Letting $\gamma \to 0$, $\delta \to 0$ and $\mu(F) \geqq \nu(F)$.

We have seen $\nu(F) = \mu(F)$ for all small compact boxes F. This implies $\nu = \mu$, and the action of G^u is uniquely ergodic.

2. Affine maps

Suppose now that A is an automorphism of G with $A(\Gamma) = \Gamma$ and $a \in G$. Then

$$T(g\Gamma) = aA(g)\Gamma$$

defines a measure preserving map T: $G/\Gamma \to \Gamma$. Then $R(g) = aA(g)a^{-1}$ is an automorphism of G and its derivative $r = dR_e$: $\mathfrak{G} \to \mathfrak{G}$ is a Lie algebra automorphism. Let $G^u(r)$ be the subgroup of G associated with the subalgebra

$\mathfrak{G}^u(f)$ corresponding to all eigenvalues λ of r with $|\lambda| > 1$. The proof in Section 1 works still in this case.

THEOREM. *Let T as above be weak mixing and assume $r \mid \mathfrak{G}^c(r)$ is a semi-simple linear map. Then the action of $G^u(r)$ on G/Γ is uniquely ergodic.*

3. Semi-simple G

Let G be a semi-simple Lie group with finite center and no compact factor and let $G = KAN$ be an Iwasawa decomposition. Then Veech [11] proved that N is uniquely ergodic on G/Γ for any co-compact discrete subgroup $\Gamma \subset G$. By the construction of an Iwasawa decomposition [5], $\mathfrak{G}^u(a) \subset \mathcal{N}$ for a in the exponential A^+ of the positive Weyl chamber and $\mathcal{N}$ the Lie algebra of N. Also ad_a is semi-simple for all $a \in A$. Thus our theorem gives Veech's result if there are any $a \in A^+$ with T_a weak mixing (if G^u_a is uniquely ergodic, so is the larger group N). The existence of such an a follows from [7] or [12].

The minimal actions of theorem1.3 of [11] (see also theor. 5, [10]) are in fact uniquely ergodic. Here one has a weak mixing R^n action; almost every element of such an action is weak mixing (apply [8] to the direct product of the action with itself.)

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